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On the stochastic porous medium equation

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Abstract

In this paper, we discuss an initial-boundary value problem and a Cauchy problem for the stochastic porous medium equation. Our basic estimates are based on the known results due to (Math. Sbornik 67 (1965) 609–642, Arch. Rational Mech. Anal. 25 (1967) 64–80). By the procedure developed in (Trans. Amer. Math. Soc. 354 (2001) 1117–1135), we obtain solutions over the given probability space rather than martingale solutions. We will also establish the existence of invariant measures when the space domain is bounded.

Keywords: Porous medium equation; Random noise; Pathwise solutions; Invariant measure

0. Introduction

In this paper, we will discuss an initial-boundary value problem and the Cauchy problem associated with the porous medium equation with random noise. The equation is of the following form:

$$u_t = \frac{1}{p-1}\Delta\left(|u|^{p-2}u\right) + \sum_{j=1}^{\infty} f_j \frac{dB_j}{dt}, \qquad p > 2, \tag{0.1}$$

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which can be also written as

$$u_t = \sum_{i=1}^{n} \partial_{x_i}\left(|u|^{p-2}\partial_{x_i}u\right) + \sum_{j=1}^{\infty} f_j \frac{dB_j}{dt}, \quad p > 2. \tag{0.2}$$

The last term represents a random noise and B_j's are the standard Brownian motions which are mutually independent. The deterministic porous medium equation has been extensively investigated. The existence of solutions to the initial-boundary value problems was proved by Dubinskii [5] and the uniqueness of solutions was proved by Raviart [13]. Their results are also presented in [10]. The equation is also a celebrated example to which the nonlinear semigroup theory of Crandall and Liggett [2] can be applied. Since the work [2] appeared, there have been numerous investigations on the properties of solutions along this line. For extensive references on the deterministic porous medium equation, see [1,6]. Meanwhile, the results of [5,13] have not received much attention in the later study. This may explain why [5,13] are not on the extensive list of references in [1] and [6].

On the other hand, nothing has been done about the stochastic porous medium equation until the work of Da Prato and Röckner [3], where martingale solutions were obtained with an additional term $\alpha\Delta u$, $\alpha > 0$ in the right-hand side, which makes the equation nondegenerate. Their approach is entirely different from ours. They first established the existence of an infinitesimal invariant measure by considering the difficult Kolmogoroff equation, which is used to obtain martingale solutions.

Here our goal is to obtain solutions to (0.1) over the given probability space. For the existence of solutions, we will follow the approach of Dubinskii [5], which consists of the Galerkin approximation and energy estimates. Even for the deterministic equation, we do not have usual estimates of approximate solutions which can guarantee the regularity that ∇u is integrable. This is due to the degenerate structure of (0.1), and causes a serious technical hurdle. It is interesting to compare our equation to the following stochastic equation.

$$u_t = \sum_{i=1}^{n} \partial_{x_i}\left(|\partial_{x_i}u|^{p-2}\partial_{x_i}u\right) + \sum_{j=1}^{\infty} f_j \frac{dB_j}{dt}, \quad p > 2. \tag{0.3}$$

This equation is also degenerate, but it is monotone in the natural function class. So the stochastic version of Minty's device can be used to obtain solutions. See [11,12]. This is not the case for (0.1). The regularity of solutions of (0.3) is definitely better than that for (0.1).

We now sketch our general strategy. For the existence of solutions, our main tool is the Galerkin approximation. The technical difficulty associated with this approximating scheme for the stochastic nonlinear equation is the lack of compactness with respect to the random variable. For Eq. (0.3), the purpose of the stochastic Minty's device is to handle this difficulty. Since our equation is not monotone, we follow the procedure initiated in [8] instead. The procedure consists in finding pathwise solutions by measuretheoretic manipulation, and proving measurability of solutions via pathwise uniqueness. For pathwise uniqueness of solutions, we follow the argument of [13]. Our existence result is in parallel to that for the deterministic equation. We also establish the existence of invariant measures by the standard procedure of Krylov–Bogolyubov [9].

When the space domain is the whole space R^n, we will obtain the solution to the Cauchy problem as the limit of the sequence of solutions to the initial-boundary value problem over the domain expanding to the whole space. We will also prove pathwise uniqueness of solution by modifying the argument in [13]. However, we have not been able to obtain estimates necessary for the existence of invariant measures, i.e., the estimates which can trap most of the energy in a bounded subset of R^n uniformly in time. In fact, such estimates can be obtained formally as long as the solution is reasonably smooth. At present, justification of formal manipulation remains open.

1. Notation and preliminaries

Let G be a bounded domain in R^n with smooth boundary. We will use the usual notation for Sobolev spaces such as $H_0^1(G)$, $H^{-q}(G)$, $W_0^{1,q}(G)$, and $W^{-1,q}(G)$, where q is a positive number. Throughout this paper, we assume

$$2 < p < \infty, \qquad \frac{1}{p'} + \frac{1}{p} = 1, \qquad r = 2 + \frac{n(p-2)}{2p}.$$

Then, $H_0^r(G)$ is a dense subspace of $L^2(G)$ and the imbedding $H_0^r(G) \to L^2(G)$ is compact. Furthermore,

$$H_0^r(G) \subset W^{2,p}(G), \qquad W^{-1,p'}(G) \subset H^{-r}(G). \tag{1.1}$$

Let us set

$$\mathcal{S}_N = \left\{ v \;\middle|\; \left\| |v|^{(p-2)/2} v \right\|_{H_0^1(G)} \leqslant N \right\}. \tag{1.2}$$

For the proof of the following facts, see [5,10,13].

Lemma 1.1. *For each $0 < N < \infty$, $\mathcal{S}_N$ is relatively compact in $L^p(G)$.*

Lemma 1.2. *If $v \in \mathcal{S}_N$, for some $0 < N < \infty$, then $|v|^{p-2}v \in W_0^{1,p'}(G)$ and*

$$\left\| |v|^{p-2} v \right\|_{W_0^{1,p'}(G)} \leqslant C N^{2(p-1)/p},$$

where C is a positive constant independent of N.

We will also use the following version of Theorem 12.1 of [10]. The proof is essentially the same.

Lemma 1.3. *Let* $\{v_m\}_{m=1}^{\infty}$ *be a sequence in* $C\big([0,T]; H_0^r(G)\big)$ *such that*

$$\big\| |v_m|^{(p-2)/2}\, v_m \big\|_{L^2(0,T;H_0^1(G))} \leqslant N, \qquad \textit{for all } m \geqslant 1,$$

for some positive constant N, *and for each* $t^* \in [0,T]$,

$$\big\| v_m(t) - v_m(t^*) \big\|_{H^{-r}(G)} \to 0$$

as $t \to t^*$ *in* $[0,T]$, *uniformly in* m *and* t^*. *Then, there is a subsequence* $\{v_{m_k}\}_{k=1}^{\infty}$ *which converges strongly in* $L^p\big(0,T; L^p(G)\big) \cap C\big([0,T]; H^{-r}(G)\big)$.

There is a positive-definite operator Λ in $L^2(G)$ such that the domain of $\Lambda^{1/2}$ is $H_0^r(G)$ and

$$\langle z,\ w\rangle_{H_0^r(G)} = \langle \Lambda^{1/2} z,\ \Lambda^{1/2} w\rangle_{L^2(G)}, \qquad \text{for all } z, w \in H_0^r(G). \tag{1.3}$$

Let $\{\phi_k\}_{k=1}^{\infty}$ be the complete set of normalized eigenfunctions of

$$\Lambda \phi_k = \lambda_k \phi_k. \tag{1.4}$$

Then, $\{\phi_k\}_{k=1}^{\infty}$ is a complete orthonormal basis for $L^2(G)$. $H_0^r(G)$ and $H^{-r}(G)$ can be characterized by

$$H_0^r(G) = \left\{ \phi = \sum_{k=1}^{\infty} a_k \phi_k \;\Big|\; \sum_{k=1}^{\infty} \lambda_k |a_k|^2 < \infty \right\} \tag{1.5}$$

and

$$H^{-r}(G) = \left\{ \phi = \sum_{k=1}^{\infty} a_k \phi_k \;\Big|\; \sum_{k=1}^{\infty} \frac{1}{\lambda_k} |a_k|^2 < \infty \right\}. \tag{1.6}$$

We denote by $\mathcal{P}_m$ the projection

$$\sum_{k=1}^{\infty} a_k \phi_k \mapsto \sum_{k=1}^{m} a_k \phi_k. \tag{1.7}$$

Obviously, $\mathcal{P}_m$ is a continuous linear operator on $L^2(G)$, $H_0^r(G)$ and $H^{-r}(G)$, respectively.

$\big(\Omega, \mathcal{F}, \mathcal{F}_t, P\big)$ is a given stochastic basis, where P is a probability measure, $\mathcal{F}$ is a σ-algebra and $\{\mathcal{F}_t\}_{t\geqslant 0}$ is a right-continuous filtration on $\big(\Omega, \mathcal{F}\big)$ such that $\mathcal{F}_0$ contains all P-negligible subsets. $\{B_j(t)\}_{j=1}^{\infty}$ is a sequence of mutually independent standard Brownian motions over $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. $E\big(\cdot\big)$ stands for expectation with respect to the probability measure P. In this paper, a stochastic integral is defined in the sense of Ito. When $\mathcal{O}$ is a topological space, $\mathcal{B}(\mathcal{O})$ denotes the Borel σ-algebra over $\mathcal{O}$. When $\mathcal{X}$ is a Banach space, an $\mathcal{X}$-valued function f is said to be $\mathcal{F}$-measurable if $f^{-1}(G) \in \mathcal{F}$ for every $G \in \mathcal{B}(\mathcal{X})$. This coincides with strong measurability for Bochner integrals when the range of f is separable. When $\mathcal{X}$ is a Banach space, $L^p(\Omega; \mathcal{X})$, $1\leqslant p < \infty$, denotes the set of all $\mathcal{X}$-valued strongly measurable functions such that

$$\int_\Omega \|f\|_{\mathcal{X}}^p \, dP < \infty.$$

An $\mathcal{X}$-valued stochastic process $Y(t)$ is said to be progressively measurable if Y restricted to the interval $[0, t]$ is $\mathcal{B}\big([0, t]\big) \otimes \mathcal{F}_t$-measurable for each $t\geqslant 0$.

Throughout this paper, we assume that for each $j\geqslant 1$, f_j is progressively measurable and $f_j \in L^2\big(\Omega; L^2(0, T; L^2(G))\big)$ such that

$$E\left(\sum_{j=1}^{\infty} \|f_j\|^2_{L^2(0,T;L^2(G))}\right) < \infty \tag{1.8}$$

for each $T > 0$. We define a continuous $L^2(G)$-valued martingale $W(t)$ by

$$W(t) = \sum_{j=1}^{\infty} \int_0^t f_j \, dB_j. \tag{1.9}$$

For general information on stochastic processes, see [7].

2. Initial-boundary value problem

We formulate the initial-boundary value problem as follows.

$$u_t = \sum_{i=1}^{n} \partial_{x_i}\big(|u|^{p-2}\partial_{x_i} u\big) + \sum_{j=1}^{\infty} f_j \frac{dB_j}{dt}, \qquad (t, x) \in (0, T) \times G, \tag{2.1}$$

$$u = 0, \qquad (t, x) \in (0, T) \times \partial G, \tag{2.2}$$

$$u(0, x) = u_0(x), \qquad x \in G. \tag{2.3}$$

Here G is a bounded domain in R^n with smooth boundary ∂G.

We adopt the following definition of a solution to (2.1)–(2.3).

Definition 2.1. Let $T > 0$ be given. An $L^2(G)$-valued process u which is progressively measurable with respect to $\{\mathcal{F}_t\}$ is called a solution of (2.1)–(2.3) if $u \in L^\infty\big(0, T; L^2(G)\big)$ and $|u|^{(p-2)/2}u \in L^2\big(0, T; H_0^1(G)\big)$ for P-almost all $\omega \in \Omega$, and if for each $\phi \in C_0^\infty(G)$,

$$\begin{aligned}\langle u(t),\ \phi\rangle - \frac{1}{p-1}\int_0^t\int_G |u(s)|^{p-2}u(s)\,\Delta\phi\,dx\,ds\\ = \langle u_0,\ \phi\rangle + \langle \phi,\ W(t)\rangle\end{aligned} \tag{2.4}$$

holds for all $t \in [0, T]$, for P-almost all $\omega \in \Omega$.

Here, $\langle\cdot\,,\ \cdot\rangle$ denotes the $L^2(G)$-product.

Theorem 2.2. *Let u_0 be $L^2(G)$-valued $\mathcal{F}_0$-measurable such that $u_0 \in L^2\big(\Omega; L^2(G)\big)$. Then, there is a unique solution to* (2.1)–(2.3) *such that for each $T > 0$,*

$$u \in L^2\big(\Omega; L^\infty(0, T; L^2(G))\big),$$

$$|u|^{(p-2)/2}u \in L^2\big(\Omega; L^2(0, T; H_0^1(G))\big)$$

and

$$\frac{\partial}{\partial t}\big(u - W\big) \in L^{p'}\big(\Omega; L^{p'}(0, T; W^{-1.p'}(G))\big).$$

We recall that an invariant measure is a probability measure such that if the probability law of the initial datum is the same as an invariant measure, then the probability law of the evolving solution is invariant in time.

Theorem 2.3. *Suppose f_j's are independent of time. There is an invariant measure over $\big(L^2(G),\ \mathcal{B}(L^2(G))\big)$ for* (2.1)–(2.2).

3. Proof of Theorem 2.2

We will use the Galerkin approximation in terms of the above basis (1.4) as presented in [5]. The purpose of the above particular basis is to obtain estimates of the time derivative of the Galerkin approximations, where the projection $\mathcal{P}_m$ plays a crucial role. We set

$$u_{m,N}(t) = c_{m,1}(t)\phi_1 + \cdots + c_{m,m}(t)\phi_m, \tag{3.1}$$

where the random functions $c_{m,i}$'s are determined from the following system of stochastic differential equations.

$$\frac{dc_{m,k}}{dt} = -\chi_N\big(\|u_{m,N}(t)\|^2_{L^2(G)}\big)\int_G |u_{m,N}(t)|^{p-2}\nabla u_{m,N}(t)\cdot\nabla\phi_k\,dx + \left\langle \phi_k, \frac{\partial W}{\partial t}\right\rangle, \qquad k=1,\dots,m, \tag{3.2}$$

$$c_{m,k}(0) = \langle u_0, \phi_k\rangle, \qquad k=1,\dots,m, \tag{3.3}$$

where χ_N belongs to $C_0^\infty(R)$ such that $\chi_N(t) = 1$ for $|t|\leqslant 2N$ and $\chi_N(t) = 0$, for $|t|\geqslant 3N$. Consider the function $g_{N,k} : R^m \to R$ defined by

$$g_{N,k}(c_1,\dots,c_m) = \chi_N\big(\|v\|^2_{L^2(G)}\big)\int_G |v|^{p-2}\nabla v\cdot\nabla\phi_k\,dx,$$

where $v = c_1\phi_1 + \cdots + c_m\phi_m$. Then, $g_{N,k}$ is globally Lipschitzian on R^m for each N, k. Hence, by the well-known theory of stochastic differential equations, we have a unique solution to (3.2)–(3.3). Let us define a stopping time by

$$\tau_N = \inf\big\{t>0\,\big|\quad \big\|u_{m,N}(t)\big\|^2_{L^2(G)}\geqslant N\big\} \tag{3.4}$$

when the set $\big\{\big\|u_{m,N}(t)\big\|^2_{L^2(G)}\geqslant N\big\}$ is nonempty, and $\tau_N=\infty$ when the set is empty. Choose any $T>0$. By Ito's formula, we have

$$\begin{aligned}\|u_{m,N}(t)\|^2_{L^2(G)} &+ 2\int_0^t \big\||u_{m,N}(s)|^{(p-2)/2}\nabla u_{m,N}(s)\big\|^2_{L^2(G)}\,ds\\ &= \|u_{m,N}(0)\|^2_{L^2(G)} + 2\sum_{j=1}^\infty\int_0^t \langle u_{m,N}(s), f_j(s)\rangle\,dB_j(s)\\ &\quad + \sum_{j=1}^\infty\int_0^t \|\mathcal{P}_m f_j(s)\|^2_{L^2(G)}\,ds\end{aligned} \tag{3.5}$$

for all $t\in[0, T\wedge\tau_N]$, for almost all ω. By the Burkholder–Davis–Gundy inequality, we have

$$E\left(\sup_{t\in[0,T\wedge\tau_N]}\left|\sum_{j=1}^\infty\int_0^t \langle u_{m,N}(s), f_j(s)\rangle\,dB_j(s)\right|\right)$$

$$
\begin{aligned}
&\leqslant CE\left(\sum_{j=1}^{\infty}\int_0^{T\wedge\tau_N}\|u_{m,N}(t)\|_{L^2(G)}^2\|f_j(t)\|_{L^2(G)}^2\,dt\right)^{1/2}\\
&\leqslant CE\left(\sup_{t\in[0,T\wedge\tau_N]}\|u_{m,N}(t)\|_{L^2(G)}\left(\sum_{j=1}^{\infty}\int_0^{T\wedge\tau_N}\|f_j(t)\|_{L^2(G)}^2\,dt\right)^{1/2}\right)\\
&\leqslant \tfrac{1}{4}E\left(\sup_{t\in[0,T\wedge\tau_N]}\|u_{m,N}(t)\|_{L^2(G)}^2\right)+CE\left(\sum_{j=1}^{\infty}\int_0^{T\wedge\tau_N}\|f_j\|_{L^2(G)}^2\,dt\right) \qquad (3.6)
\end{aligned}
$$

for some positive constant C independent of m, N and $T > 0$. Thus, it follows from (3.5) that

$$
\begin{aligned}
&E\left(\sup_{t\in[0,T\wedge\tau_N]}\|u_{m,N}(t)\|_{L^2(G)}^2\right)+4E\left(\int_0^{T\wedge\tau_N}\big\||u_{m,N}(t)|^{(p-2)/2}\nabla u_{m,N}(t)\big\|_{L^2(G)}^2\,dt\right)\\
&\quad\leqslant 2E\big(\|u_m(0)\|_{L^2(G)}^2\big)+C\sum_{j=1}^{\infty}E\left(\int_0^{T\wedge\tau_N}\|f_j(s)\|_{L^2(G)}^2\,ds\right), \qquad (3.7)
\end{aligned}
$$

for some positive constant C independent of m, N and $T > 0$. Next we see that for $N_1 < N_2$,

$$\tau_{N_1}\leqslant\tau_{N_2} \qquad (3.8)$$

holds for almost all ω. By the pathwise uniqueness of solutions,

$$u_{m,N_1}\equiv u_{m,N_2} \qquad \text{on } [0,\ \tau_{N_1}\wedge\tau_{N_2}],\ \text{ for almost all } \omega. \qquad (3.9)$$

We define for almost all ω,

$$\tau_\infty=\lim_{N\to\infty}\tau_N \qquad (3.10)$$

and

$$u_m(t)=u_{m,N}(t) \qquad \text{for } t\in[0,T\wedge\tau_N]. \qquad (3.11)$$

By virtue of (3.7), we have

$$P\big(\{\tau_N<T\}\big)\leqslant M/N. \qquad (3.12)$$

Since $\{\tau_\infty < T\} \subset \bigcap_{N=1}^{\infty}\{\tau_N < T\}$,

$$P\big(\{\tau_\infty < T\}\big) = 0. \tag{3.13}$$

Hence, u is defined on $[0, T)$, for almost all ω. Since T was arbitrarily chosen, for each $T > 0$, u_m exists on $[0, T]$, for almost all ω. It follows from (3.7)–(3.13) and Fatou's lemma that for each $T > 0$,

$$\begin{aligned} &E\bigg(\sup_{t\in[0,T]} \|u_m(t)\|^2_{L^2(G)}\bigg) + 4E\bigg(\int_0^T \big\| |u_m(t)|^{(p-2)/2}\nabla u_m(t)\big\|^2_{L^2(G)}\,dt\bigg) \\ &\quad\leqslant 2E\big(\|u_m(0)\|^2_{L^2(G)}\big) + C\sum_{j=1}^{\infty} E\bigg(\int_0^T \|f_j(s)\|^2_{L^2(G)}\,ds\bigg). \end{aligned} \tag{3.14}$$

It also follows from (3.2), (3.3) and (3.5) that

$$\begin{aligned} &\langle u_m(t),\ \phi_k\rangle - \frac{1}{p-1}\int_0^t\int_G |u_m(s)|^{p-2}u_m(s)\,\Delta\phi_k\,dx\,ds \\ &\quad= \langle u_0,\ \phi_k\rangle + \langle \phi_k,\ W(t)\rangle, \quad k = 1,\ldots,m \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\|u_m(t)\|^2_{L^2(G)} + 2\int_0^t \big\| |u_m(s)|^{(p-2)/2}\nabla u_m(s)\big\|^2_{L^2(G)}\,ds \\ &\quad= \|u_m(0)\|^2_{L^2(G)} + 2\sum_{j=1}^{\infty}\int_0^t \langle u_m(s),\ f_j(s)\rangle\,dB_j(s) \\ &\qquad+ \sum_{j=1}^{\infty}\int_0^t \|\mathcal{P}_m f_j(s)\|^2_{L^2(G)}\,ds \end{aligned} \tag{3.16}$$

for all t, for almost all $\omega \in \Omega$. By (1.1) and Lemma 1.2, we find

$$\Delta\big(|u_m|^{p-2}u_m\big) \in L^{p'}\big(0, T; H^{-r}(G)\big) \tag{3.17}$$

for almost all ω. Hence, it follows from (1.6), (1.7) and (3.15) that

$$\frac{\partial}{\partial t}\big(u_m - \mathcal{P}_m W\big) = \frac{1}{p-1}\,\mathcal{P}_m\Delta\big(|u_m|^{p-2}u_m\big) \in L^{p'}\big(0, T; H^{-r}(G)\big) \tag{3.18}$$

for almost all ω. Hence, we have

$$E\left(\left\|\frac{\partial}{\partial t}\big(u_m-\mathcal{P}_mW\big)\right\|^{p'}_{L^{p'}(0,T;H^{-r}(G))}\right)\leqslant CE\left(\int_0^T\left\||u_m(t)|^{(p-2)/2}\nabla u_m(t)\right\|^2_{L^2(G)}dt\right)$$
$$\leqslant CE\big(\|u_m(0)\|^2_{L^2(G)}\big)+C\sum_{j=1}^{\infty}E\left(\int_0^T\|f_j(s)\|^2_{L^2(G)}ds\right). \tag{3.19}$$

for all $m\geqslant 1$, where C denotes some positive constants independent of m and T.

Let us choose any $T>0$ and write for each $m\geqslant 1$,

$$Q_m=\sup_{t\in[0,T]}\left\|u_m(t)\right\|^2_{L^2(G)}+\int_0^T\left\||u_m|^{(p-2)/2}\nabla u_m\right\|^2_{L^2(G)}dt$$
$$+\left\|\frac{\partial}{\partial t}\big(u_m-\mathcal{P}_mW\big)\right\|^{p'}_{L^{p'}(0,T;H^{-r}(G))}. \tag{3.20}$$

It follows from (3.14) and (3.19) that

$$P\left(\bigcap_{L=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{m=k}^{\infty}\Big\{Q_m\geqslant L\Big\}\right)=0. \tag{3.21}$$

Thus, there is a subset $\tilde\Omega$ such that

$$\tilde\Omega\subset\bigcup_{L=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{m=k}^{\infty}\Big\{Q_m\leqslant L\Big\}, \tag{3.22}$$

$$P\big(\Omega\setminus\tilde\Omega\big)=0 \tag{3.23}$$

and such that for each $\omega\in\tilde\Omega$,

$$W\in C\big([0,T];L^2(G)\big) \tag{3.24}$$

and

$$\text{Eqs. (3.15) and (3.16) hold for all } t\in[0,T] \text{ and all } m\geqslant 1. \tag{3.25}$$

We note that for each $\omega\in\tilde\Omega$, there is a subsequence $\{u_{m_k}\}_{k=1}^{\infty}$ depending on ω such that

$$\|u_{m_k}(t)-u_{m_k}(t^*)\|_{H^{-r}(G)}\to 0\qquad\text{as } t\to t^* \text{ in } [0,T] \tag{3.26}$$

uniformly in k and t^*. This follows from (3.24) and

$$\left\| \frac{\partial}{\partial t}\big(u_{m_k} - \mathcal{P}_{m_k} W\big) \right\|^{p'}_{L^{p'}(0,T;H^{-r}(G))} \leqslant Q_{m_k} \leqslant L_\omega \tag{3.27}$$

for all k, for some constant L_ω. By virtue of Lemma 1.3, we can further extract a subsequence still denoted by $\{u_{m_k}\}_{k=1}^{\infty}$ depending on ω such that

$$\text{Each } u_{m_k} \text{ belongs to } C\big([0,T]; H_0^r(G)\big), \tag{3.28}$$

$$u_{m_k} \to u \qquad \text{strongly in } C\big([0,T]; H^{-r}(G)\big), \tag{3.29}$$

$$u_{m_k} \to u \qquad \text{weak star in } L^{\infty}\big(0,T; L^2(G)\big), \tag{3.30}$$

$$u_{m_k} \to u \qquad \text{strongly in } L^p\big(0,T; L^p(G)\big) \tag{3.31}$$

and

$$u_{m_k}(t) \to u(t) \qquad \text{weakly in } L^2(G) \text{ for each } t \in [0,T], \tag{3.32}$$

for some function u. It follows that

$$|u_{m_k}|^{(p-2)/2} u_{m_k} \to |u|^{(p-2)/2} u \quad \text{weakly in } L^2\big(0,T; H_0^1(G)\big). \tag{3.33}$$

Since $u \in C\big([0,T]; H^{-r}(G)\big)$, (3.29) and (3.30) imply that

$$u(t_2) \to u(t_1) \qquad \text{weakly in } L^2(G) \text{ as } t_2 \to t_1 \text{ in } [0,T] \tag{3.34}$$

and

$$u \in C([0,T]; H^{-\varepsilon}(G)), \qquad \text{for every } \varepsilon > 0. \tag{3.35}$$

It also follows from (3.25), (3.31) and (3.32) that

$$\begin{aligned} &\langle u(t),\ \phi\rangle - \frac{1}{p-1}\int_0^t \int_G |u(s)|^{p-2} u(s)\, \Delta\phi\, dx\, ds \\ &\quad = \langle u_0,\ \phi\rangle + \langle \phi,\ W(t)\rangle, \end{aligned} \tag{3.36}$$

for all $\phi \in H_0^r(G)$ and all $t \in [0,T]$. Hence, (2.4) is satisfied. Furthermore, by (3.16) and (3.31)–(3.33), we find that

$$\begin{aligned} &\|u(t)\|^2_{L^2(G)} + 2\int_0^t \big\||u(s)|^{(p-2)/2}\nabla u(s)\big\|^2_{L^2(G)}\,ds \\ &\quad\leqslant \|u_0\|^2_{L^2(G)} + 2\sum_{j=1}^{\infty}\int_0^t \langle u(s), f_j(s)\rangle\,dB_j(s) \\ &\qquad + \sum_{j=1}^{\infty}\int_0^t \|f_j(s)\|^2_{L^2(G)}\,ds \end{aligned} \tag{3.37}$$

holds for all $t \in [0,T]$, for almost all ω. Hence,

$$\overline{\lim_{t\to 0}}\, \|u(t)\|_{L^2(G)} \leqslant \|u_0\|_{L^2(G)} \tag{3.38}$$

for almost all ω and by (3.34)

$$u(t) \to u_0 \qquad \text{strongly in } L^2(G), \text{ as } t \downarrow 0, \tag{3.39}$$

for almost all ω. We now consider pathwise uniqueness of solutions according to the argument in [13]. For fixed $\omega \in \tilde{\Omega}$, suppose that u and v satisfy (3.36), and

$$|u|^{(p-2)/2}u,\ |v|^{(p-2)/2}v \in L^2\big(0,T; H_0^1(G)\big), \tag{3.40}$$

$$u,\ v \in L^\infty\big(0,T; L^2(G)\big). \tag{3.41}$$

Then, it holds that

$$\begin{aligned} &\left\langle \frac{\partial}{\partial t}\big(u(t)-v(t)\big),\ \phi\right\rangle \\ &\quad = \frac{1}{p-1}\int_G \big(|u(t)|^{p-2}u(t) - |v(t)|^{p-2}v(t)\big)\,\Delta\phi\,dx \end{aligned} \tag{3.42}$$

for all $\phi \in H_0^r(G)$ in the sense of distribution over $(0,T)$. Since (3.42) implies

$$\frac{\partial}{\partial t}\big(u-v\big) = \frac{1}{p-1}\Delta\big(|u|^{p-2}u - |v|^{p-2}v\big)$$

in the sense of distribution over $(0, T) \times G$, it follows that

$$\frac{\partial}{\partial t}(u - v) \in L^{p'}\big(0, T; W^{-1,p'}(G)\big). \tag{3.43}$$

Hence, we may take $\phi = \Delta^{-1}(u - v) \in L^p\big(0, T; W_0^{1,p}(G)\big)$ in (3.42) according to Lemma 1.3 in [13], and obtain

$$\|u(t) - v(t)\|^2_{H^{-1}(G)} \leqslant 0 \tag{3.44}$$

for each $t \in [0, T]$ so that $u \equiv v$. Here Δ^{-1} is the inverse of the Laplacian with zero Dirichlet boundary condition.

Next choose any closed ball V in $H^{-r}(G)$, and write

$$V_\nu = \left\{ z \in H^{-r}(G) \,\Big|\, \|z - y\|_{H^{-r}(G)} \leqslant \frac{1}{\nu}, \quad \text{for some } y \in V \right\}. \tag{3.45}$$

We also write

$$\begin{aligned} Q_{m,t} = {} & \sup_{s\in[0,t]} \big\|u_m(s)\big\|^2_{L^2(G)} + \int_0^t \big\||u_m|^{(p-2)/2}\nabla u_m\big\|^2_{L^2(G)}\,ds \\ & + \left\|\frac{\partial}{\partial s}\big(u_m - \mathcal{P}_m W\big)\right\|^{p'}_{L^{p'}(0,t;H^{-r}(G))}. \end{aligned} \tag{3.46}$$

Then, for each fixed $t^* \in (0, T]$, we will show that

$$\tilde{\Omega} \bigcap \big\{u(t^*) \in V\big\} = \tilde{\Omega} \bigcap \left(\bigcup_{L=1}^{\infty} \bigcap_{\nu=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \big\{u_m(t^*) \in V_\nu \quad \text{and} \quad Q_{m,t^*} \leqslant L\big\} \right). \tag{3.47}$$

Suppose ω^* belongs to the left-hand side. According to the construction of u for each $\omega \in \tilde{\Omega}$, there is a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ such that

$$Q_{m_j,t^*} \leqslant Q_{m_j} \leqslant L, \qquad \text{for all } j \geqslant 1 \tag{3.48}$$

for some $L \geqslant 1$, and $u_{m_j} \to u$ in the sense of (3.29)–(3.33). Thus, ω^* belongs to the right-hand side. Next suppose ω^* belongs to the right-hand side. Then, there is some $1 \leqslant L < \infty$ such that

$$\omega^* \in \tilde{\Omega} \bigcap \bigcap_{\nu=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \big\{u_m(t^*) \in V_\nu \quad \text{and} \quad Q_{m,t^*} \leqslant L\big\}.$$

Fix any $\nu \geqslant 1$. There is a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ which satisfies

$$u_{m_j}(t^*) \in V_\nu, \qquad Q_{m_j,t^*} \leqslant L \tag{3.49}$$

for all $j \geqslant 1$. Hence, we can further extract a subsequence still denoted by $\{u_{m_j}\}_{j=1}^{\infty}$ such that $u_{m_j} \to u^*$ for some function u^* in the sense of (3.29)–(3.33) over the interval $[0, t^*]$. Since (3.15) holds for all $t \in [0, T]$ and $1 \leqslant k \leqslant m$, for every $m \geqslant 1$, for each $\omega \in \tilde{\Omega}$, we have

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}\big(u(t) - u^*(t)\big),\ \phi \right\rangle \\
&\quad = \frac{1}{p-1} \int_G \big(|u(t)|^{p-2}u(t) - |u^*(t)|^{p-2}u^*(t)\big)\, \Delta\phi\, dx
\end{aligned} \tag{3.50}$$

for all $\phi \in H_0^r(G)$ in the sense of distribution over $(0, t^*)$. By the same argument as above, we have $u \equiv u^*$ on the interval $[0, t^*]$. Thus, $u(t^*) \in V_\nu$. Since this is true for all $\nu \geqslant 1$, $u(t^*) \in V$, and ω^* belongs to the left-hand side. So (3.47) holds.

Next we note that the mapping

$$(c_{m,1}, \ldots, c_{m,m}) \mapsto \big(u_m(t^*),\ Q_{m,t^*}\big)$$

is continuous from $\big(C([0, t^*])\big)^m$ into $H^{-r}(G) \times R$, where we use (3.18), and that $c_{m,k}$'s are progressively measurable. Thus, the right-hand side of (3.47) is $\mathcal{F}_{t^*}$-measurable, and $\big\{u(t^*) \in V\big\} \in \mathcal{F}_{t^*}$. It follows that $\big\{u(t) \in F\big\} \in \mathcal{F}_t$ for each Borel subset F of $H^{-r}(G)$ and each $t \in [0, T]$. Since $u \in C\big([0, T]; H^{-r}(G)\big)$, for almost all ω, $\big\{(t, \omega)\big| 0 \leqslant t \leqslant t^*,\ u(t, \omega) \in F\big\} \in \mathcal{B}\big([0, T]\big) \otimes \mathcal{F}_{t^*}$, for each $F \in \mathcal{B}\big(H^{-1}(G)\big)$ and each $t^* \in (0, T]$. Hence, it is also true for each $F \in \mathcal{B}\big(L^2(G)\big)$. So u is $L^2(G)$-valued progressively measurable.

Next we set

$$\begin{aligned}
Q = \sup_{t \in [0,T]} \|u(t)\|_{L^2(G)}^2 &+ \int_0^T \big\| |u|^{(p-2)/2} \nabla u \big\|_{L^2(G)}^2\, dt \\
&+ \left\| \frac{\partial}{\partial t}\big(u - W\big) \right\|_{L^{p'}(0,T; H^{-r}(G))}^{p'}.
\end{aligned} \tag{3.51}$$

We will show that for each $\omega \in \tilde{\Omega}$,

$$Q \wedge K \leqslant \varliminf_{m \to \infty} Q_m \wedge K \tag{3.52}$$

for all positive constant K. Fix any $\omega \in \tilde{\Omega}$ and $K > 0$. If $\underline{\lim}_{m\to\infty} Q_m \wedge K = K$, the inequality holds. Suppose that $\underline{\lim}_{m\to\infty} Q_m \wedge K = L < K$. Then, for each $\varepsilon > 0$, there is a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ such that

$$Q_{m_j} \leqslant L + \varepsilon, \qquad \text{for all } j \geqslant 1. \tag{3.53}$$

Hence, there is a subsequence still denoted by $\{u_{m_j}\}_{j=1}^{\infty}$ such that $u_{m_j} \to u^*$ for some function in the sense of (3.29)–(3.33). Again by the pathwise uniqueness, $u \equiv u^*$ and

$$Q \leqslant L + \varepsilon \tag{3.54}$$

holds. Since $\varepsilon > 0$ is arbitrary, (3.52) is valid. By (3.14), (3.19) and Fatou's lemma, we have

$$E\big(Q \wedge K\big) \leqslant \underline{\lim}\, E\big(Q_m \wedge K\big) \leqslant C E\big(\|u_0\|^2_{L^2(G)}\big) + C \sum_{j=1}^{\infty} E\bigg(\int_0^T \|f_j(t)\|^2_{L^2(G)}\, dt\bigg) \tag{3.55}$$

for some positive constant C independent of $T > 0$. By passing $K \uparrow \infty$, we arrive at

$$\begin{aligned} &E\bigg(\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(G)}\bigg) + E\bigg(\int_0^T \big\||u|^{(p-2)/2} \nabla u\big\|^2_{L^2(G)}\, dt\bigg) \\ &\quad \leqslant C E\big(\|u_0\|^2_{L^2(G)}\big) + C \sum_{j=1}^{\infty} E\bigg(\int_0^T \|f_j(t)\|^2_{L^2(G)}\, dt\bigg). \end{aligned} \tag{3.56}$$

Since it holds that

$$\frac{\partial}{\partial t}\big(u - W\big) = \Delta\big(|u|^{p-2} u\big)$$

in the sense of distribution over $(0, T) \times G$, for almost all ω, it follows from (3.56) that

$$\begin{aligned} &E\bigg(\bigg\|\frac{\partial}{\partial t}\big(u - W\big)\bigg\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(G))}\bigg) \\ &\quad \leqslant C E\big(\|u_0\|^2_{L^2(G)}\big) + C \sum_{j=1}^{\infty} E\bigg(\int_0^T \|f_j(t)\|^2_{L^2(G)}\, dt\bigg). \end{aligned} \tag{3.57}$$

This completes the proof of Theorem 2.2.

4. Proof of Theorem 2.3

We first note that the above existence result is valid when the initial time is any $s > 0$, and the initial function u_s is $L^2(G)$-valued $\mathcal{F}_s$-measurable such that $u_s \in L^2\big(\Omega; L^2(G)\big)$. Let us denote by $X(t; s, z)$ the solution which satisfies the initial condition $u(s) = z \in L^2(G)$. We define

$$\mathcal{P}(s, z; t, \Gamma) = P\Big(\big\{X(t; s, z) \in \Gamma\big\}\Big)$$

for $0 \leqslant s \leqslant t$, $z \in L^2(G)$ and $\Gamma \in \mathcal{B}(L^2(G))$. Then, we have the following properties of the Markov transition function.

(I) For each $0 \leqslant s < t$ and $z \in L^2(G)$, $\mathcal{P}(s, z; t, \cdot)$ is a probability measure over $\big(L^2(G), \mathcal{B}(L^2(G))\big)$.

(II) $\mathcal{P}(s, \cdot; t, \Gamma)$ is $\mathcal{B}(L^2(G))$-measurable.

(III) For any $0 \leqslant s < t$, $h > 0$, $z \in L^2(G)$ and $\Gamma \in \mathcal{B}(L^2(G))$, it holds that

$$\mathcal{P}(s + h, z; t + h, \Gamma) = \mathcal{P}(s, z; t, \Gamma).$$

(IV) $\mathcal{P}(0, z; \cdot, \Gamma)$ is $\mathcal{B}([0, \infty))$-measurable.

(V) For each $0 \leqslant s \leqslant \eta \leqslant t$, and each $z \in L^2(G)$, $\Gamma \in \mathcal{B}(L^2(G))$,

$$\mathcal{P}(s, z; t, \Gamma) = \int_{L^2(G)} \mathcal{P}(s, z; \eta, dy)\mathcal{P}(\eta, y; t, \Gamma).$$

The property (I) is obvious. We show (II). Choose any $0 \leqslant s < t$, and $z_1, z_2 \in L^2(G)$. By the same argument as for (3.44), we find

$$\big\|X(t; s, z_2) - X(t; s, z_1)\big\|^2_{H^{-1}(G)} \leqslant \big\|z_2 - z_1\big\|^2_{H^{-1}(G)} \tag{4.1}$$

for almost all ω. Let ψ be a bounded continuous function on $L^2(G)$. Define

$$\psi_m(y) = \psi(\mathcal{P}_m y), \tag{4.2}$$

where $\mathcal{P}_m$ is the projection defined by (1.7). Then, ψ_m is a bounded continuous function on $H^{-1}(G)$, and, for each $z \in L^2(G)$,

$$\psi_m(z) \to \psi(z) \qquad \text{as } m \to \infty. \tag{4.3}$$

Hence, $E\big(\psi_m(X(t; s, z)\big)$ is continuous in $z \in L^2(G)$ for each $m \geqslant 1$, and

$$E\big(\psi_m(X(t; s, z)\big) \to E\big(\psi(X(t; s, z)\big) \quad \text{as } m \to \infty. \tag{4.4}$$

Thus, $E\big(\psi(X(t; s, \cdot)\big)$ is $\mathcal{B}(L^2(G))$-measurable. This implies (II).

For (III), choose any $0 \leqslant s_1 < s_2$ and $z \in L^2(G)$. Let u_1 be the solution satisfying the condition $u_1(s_1) = z$, and u_2 be the solution satisfying $u_2(s_2) = z$. Let us set

$$u(t) = u_1(s_1 + t), \qquad \hat{u}(t) = u_2(s_2 + t), \qquad t \geqslant 0$$

and

$$W_\dagger(t) = W(t + s_1) - W(s_1), \qquad \hat{W}_\dagger(t) = W(t + s_2) - W(s_2), \qquad t \geqslant 0.$$

Then, u and $\hat{u}$ are solutions of (2.1)–(2.3) with $u_0 = z$ and W replaced by $W_\dagger$ and $\hat{W}_\dagger$, respectively. We want to show that for any $t^* > 0$ and $\Gamma \in \mathcal{B}\big(L^2(G)\big)$,

$$P\Big(\big\{u(t^*) \in \Gamma\big\}\Big) = P\Big(\big\{\hat{u}(t^*) \in \Gamma\big\}\Big). \tag{4.5}$$

If there were subsequences $\{u_{m_j}\}_{j=1}^{\infty}$, $\{\hat{u}_{m_j}\}_{j=1}^{\infty}$ constructed through the scheme (3.1)–(3.3) such that $u_{m_j} \to u$ and $\hat{u}_{m_j} \to \hat{u}$ in some sense for almost all ω, then (4.5) would follow because $W_\dagger$ and $\hat{W}_\dagger$ have the same probability law. However, this is not the case according to our construction of u and $\hat{u}$. Recall that the choice of a convergent subsequence to obtain $u(\omega)$ depended on ω. So we proceed differently. As in (3.47), we have

$$\tilde{\Omega} \bigcap \big\{u(t^*) \in V\big\} = \tilde{\Omega} \bigcap \left(\bigcup_{L=1}^{\infty} \bigcap_{v=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \big\{u_m(t^*) \in V_v \quad \text{and} \quad Q_{m,t^*} \leqslant L\big\} \right) \tag{4.6}$$

and

$$\tilde{\Omega} \bigcap \big\{\hat{u}(t^*) \in V\big\} = \tilde{\Omega} \bigcap \left(\bigcup_{L=1}^{\infty} \bigcap_{v=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \big\{\hat{u}_m(t^*) \in V_v \quad \text{and} \quad \hat{Q}_{m,t^*} \leqslant L\big\} \right), \tag{4.7}$$

where V is a closed ball in $H^{-r}(G)$ and V_v is defined by (3.45). Here $\tilde{\Omega}$ has been modified by a P-negligible set to accommodate both u and $\hat{u}$, and Q_{m,t^*} and $\hat{Q}_{m,t^*}$ are defined as in (3.46) on the interval $[0, t^*]$ for u and $\hat{u}$, respectively. For given $\varepsilon > 0$, there are some $L^* \geqslant 1$, $v^* \geqslant 1$, $k^* \geqslant 1$ and $N \geqslant 1$ such that

$$\begin{aligned} P\Bigg(&\left(\bigcup_{L=1}^{\infty} \bigcap_{v=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \big\{u_m(t^*) \in V_v \quad \text{and} \quad Q_{m,t^*} \leqslant L\big\} \right) \\ &\Delta \left(\bigcup_{m=k^*}^{N} \big\{u_m(t^*) \in V_{v^*} \quad \text{and} \quad Q_{m,t^*} \leqslant L^*\big\} \right)\Bigg) < \varepsilon \end{aligned} \tag{4.8}$$

and

$$P\left(\left(\bigcup_{L=1}^{\infty}\bigcap_{\nu=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{m=k}^{\infty}\left\{\hat{u}_m(t^*)\in V_\nu \quad \text{and} \quad \hat{Q}_{m,t^*}\leqslant L\right\}\right)\right.$$
$$\left.\Delta\left(\bigcup_{m=k^*}^{N}\left\{\hat{u}_m(t^*)\in V_{\nu^*} \quad \text{and} \quad \hat{Q}_{m,t^*}\leqslant L^*\right\}\right)\right) < \varepsilon, \tag{4.9}$$

where Δ is the usual symbol for the set difference, i.e., $A\Delta B = (A\setminus B)\cup(B\setminus A)$. As in (3.1), let $c_{m,j}$, $\hat{c}_{m,j}$, $1\leqslant j\leqslant m$, $1\leqslant m\leqslant N$ be the coefficients for u_m and $\hat{u}_m$, respectively. By the well-known fact from stochastic differential equations, the joint probability distribution of $\{c_{m,j}\}_{1\leqslant j\leqslant m}^{1\leqslant m\leqslant N}$ is the same as that of $\{\hat{c}_{m,j}\}_{1\leqslant j\leqslant m}^{1\leqslant m\leqslant N}$. As in Section 3, the mapping

$$(c_{m,1},\ldots,c_{m,m})\mapsto\left(u_m(t^*),\ Q_{m,t^*}\right)$$

is continuous from $\left(C([0,t^*])\right)^m$ into $H^{-r}(G)\times R$. This implies

$$P\left(\bigcup_{m=k^*}^{N}\left\{u_m(t^*)\in V_{\nu^*} \quad \text{and} \quad Q_{m,t^*}\leqslant L^*\right\}\right)$$
$$= P\left(\bigcup_{m=k^*}^{N}\left\{\hat{u}_m(t^*)\in V_{\nu^*} \quad \text{and} \quad \hat{Q}_{m,t^*}\leqslant L^*\right\}\right),$$

which yields (4.5).

For (IV), it is enough to show that for each bounded continuous function ψ on $L^2(G)$, $E\big(\psi\big(X(\cdot\,;0,z)\big)$ is $\mathcal{B}([0,\infty))$-measurable. As above, we define ψ_m by (4.2). Since we have $X(\cdot\,;0,z)\in C\big([0,\infty);H^{-r}(G)\big)$, for almost all ω, $E\big(\psi_m(X(t;0,z))\big)$ is continuous in t. But for each t, $E\big(\psi_m(X(t;0,z))\big)\to E\big(\psi(X(t;0,z))\big)$, and thus $E\big(\psi(X(\cdot\,;0,z))\big)$ is $\mathcal{B}([0,\infty))$-measurable.

The Chapman–Kolmogoroff property (V) follows from the following Markov property.

Lemma 4.1. *For any $0\leqslant s\leqslant\eta\leqslant t<\infty$, and any bounded continuous function ψ on $L^2(G)$,*

$$E\Big(\psi\big(X(t;s,z)\big)\Big|\,\mathcal{F}_\eta\Big) = P_{\eta,t}(\psi)\big(X(\eta;s,z)\big), \tag{4.10}$$

where the operator $P_{\eta,t}$ is defined by

$$P_{\eta,t}(\psi)(y) = E\Big(\psi\big(X(t;\eta,y)\big)\Big).$$

Proof. We simply reproduce the argument [4, p. 249–250] with some technical modification. By the pathwise uniqueness of solutions, we have

$$X(t;s,z)=X(t;\eta,X(\eta;s,z)) \qquad \text{for almost all } \omega.$$

By setting $\xi=X(\eta;s,z)$, (4.10) can be written as

$$E\left(\psi\big(X(t;s,\xi)\big)\Big|\mathcal{F}_\eta\right)=P_{\eta,t}(\psi)(\xi) \tag{4.11}$$

for almost all ω. By the argument in [4, p. 250], (4.11) is valid when ξ is an $L^2(G)$-valued $\mathcal{F}_\eta$-measurable simple function. For $\xi=X(\eta;s,z)$, there is a sequence of such simple functions $\{\xi_k\}_{k=1}^\infty$ such that $\xi_k\to\xi$ in $L^2\big(\Omega;L^2(G)\big)$. As above, we define ψ_m by (4.2). It follows from (4.1) that

$$E\left(\|X(t;\eta,\xi_k)-X(t;\eta,\xi_l)\|^2_{H^{-1}(G)}\right)\leqslant E\left(\|\xi_k-\xi_l\|^2_{H^{-1}(G)}\right) \tag{4.12}$$

and thus, there is a subsequence $\{\xi_{k_j}\}_{j=1}^\infty$ such that

$$\xi_{k_j}\to\xi \quad \text{in } L^2(G), \qquad \psi_m\big(X(t;\eta,\xi_{k_j})\big)\to\psi_m\big(X(t;\eta,\xi)\big)$$

for almost all ω. So (4.11) holds for each ψ_m. We then pass $m\to\infty$ to arrive at (4.10). $\square$

We are now ready to establish the existence of an invariant measure by the Krylov–Bogoliubov procedure. Fix any $z\in L^2(G)$ and set

$$\mu_T(\Gamma)=\frac{1}{T}\int_0^T\mathcal{P}(0,z;t,\Gamma)\,dt \tag{4.13}$$

for each $\Gamma\in\mathcal{B}(L^2(G))$. Then, μ_T is well-defined by (IV) and is a probability measure over $\big(L^2(G),\mathcal{B}(L^2(G))\big)$. Since f_j's are independent of time variable, it follows from (3.56) that

$$E\left(\frac{1}{T}\int_0^T\big\||u|^{(p-2)/2}\nabla u\big\|^2_{L^2(G)}\,dt\right)\leqslant M \tag{4.14}$$

for all $T>0$, for some positive constant M. Hence, for any $\varepsilon>0$, there is some $0<L<\infty$ such that

$$\frac{1}{T}\int_0^T P\left\{\big\||u|^{(p-2)/2}\nabla u\big\|^2_{L^2(G)}\geqslant L\right\}dt<\varepsilon \tag{4.15}$$

for all $T > 0$, and thus, for some $0 < N < \infty$,

$$\mu_T\{\mathcal{S}_N\} > 1 - \varepsilon \tag{4.16}$$

for all $T > 0$, where $\mathcal{S}_N$ is defined by (1.2). By virtue of Lemma 1.1, the family of probability measures $\{\mu_T\}$ is tight. So there is a weakly convergent sequence $\{\mu_{T_k}\}_{j=1}^{\infty}$. Let μ be its weak limit, and choose any bounded continuous function ϕ on $H^{-1}(G)$. Then, by (4.1), the function

$$\int_{L^2(G)} \mathcal{P}(0,\ \cdot\,; t, dw)\phi(w)$$

is bounded and continuous on $L^2(R)$. Hence, we have

$$\begin{aligned}
&\lim_{k\to\infty} \int_{L^2(G)} \mu_{T_k}(dy) \int_{L^2(G)} \mathcal{P}(0, y; t, dw)\phi(w) \\
&\quad = \int_{L^2(G)} \mu(dy) \int_{L^2(G)} \mathcal{P}(0, y; t, dw)\phi(w)
\end{aligned} \tag{4.17}$$

With help of (II)–(V), we also find that

$$\begin{aligned}
&\int_{L^2(G)} \mu_{T_k}(dy) \int_{L^2(G)} \mathcal{P}(0, y; t, dw)\phi(w) \\
&\quad = \frac{1}{T_k}\int_0^{T_k} \left(\int_{L^2(G)} \mathcal{P}(0, z; s, dy) \int_{L^2(G)} \mathcal{P}(0, y; t, dw)\phi(w) \right) ds \\
&\quad = \frac{1}{T_k}\int_0^{T_k} \left(\int_{L^2(G)} \mathcal{P}(0, z; s+t, dy)\phi(y) \right) ds \\
&\quad = \frac{1}{T_k}\int_t^{T_k+t} \left(\int_{L^2(G)} \mathcal{P}(0, z; \eta, dy)\phi(y) \right) d\eta.
\end{aligned} \tag{4.18}$$

But we have

$$\lim_{k\to\infty} \left| \frac{1}{T_k}\int_t^{T_k+t} \left(\int_{L^2(G)} \mathcal{P}(0, z; \eta, dy)\phi(y) \right) d\eta - \int_{L^2(G)} \mu_{T_k}(dy)\, \phi(y) \right| = 0 \tag{4.19}$$

and since ϕ is also a bounded continuous function on $L^2(G)$,

$$\lim_{k\to\infty} \int_{L^2(G)} \mu_{T_k}(dy)\, \phi(y) = \int_{L^2(G)} \mu(dy)\, \phi(y). \tag{4.20}$$

Therefore, it holds that

$$\int_{L^2(G)} \mu(dy) \int_{L^2(G)} \mathcal{P}(0, y; t, dw)\phi(w) = \int_{L^2(G)} \mu(dy)\phi(y). \tag{4.21}$$

Next let ψ be a bounded continuous function on $L^2(G)$, and let $\psi_m(y)$ be defined by (4.2). Then, each ψ_m is a bounded continuous function on $H^{-1}(G)$. Since (4.21) holds for each $\phi = \psi_m$, it holds for $\phi = \psi$ by the bounded convergence theorem. Hence, μ is an invariant measure. This concludes the proof of Theorem 2.3.

5. The Cauchy problem in R^n

In this section, we assume that the initial function u_0 is $L^2(R^n)$-valued $\mathcal{F}_0$-measurable such that

$$E\big(\|u_0\|^p_{L^2(R^n)}\big) < \infty \tag{5.1}$$

and f_j's are $L^2(R^n)$-valued progressively measurable such that

$$E\left(\sum_{j=1}^{\infty} \int_0^T \|f_j\|^2_{L^2(R^n)}\, dt\right)^{p/2} < \infty \tag{5.2}$$

for each $T > 0$. We then consider the initial value problem.

$$u_t = \sum_{i=1}^{n} \partial_{x_i}\big(|u|^{p-2}\partial_{x_i} u\big) + \sum_{j=1}^{\infty} f_j \frac{dB_j}{dt}, \quad (t, x) \in (0, T) \times R^n \tag{5.3}$$

$$u(0, x) = u_0(x), \qquad x \in R^n. \tag{5.4}$$

A solution is still defined by Definition 2.1 above with G replaced by R^n.

Theorem 5.1. *Under the conditions* (5.1) *and* (5.2), *there is a pathwise unique solution to* (5.3)–(5.4) *such that for each* $T > 0$,

$$u \in L^2\big(\Omega; L^\infty(0, T; L^2(R^n))\big),$$

$$|u|^{(p-2)/2}u \in L^2\big(\Omega; L^2(0, T; H^1(R^n))\big)$$

and

$$\frac{\partial}{\partial t}\big(u - W\big) \in L^{p'}\big(\Omega; L^{p'}(0, T; W^{-1.p'}(R^n))\big).$$

The proof of this result will be given in the remainder of this section. The general strategy is to obtain the solution as a limit of the sequence of solutions of the initial-boundary value problem over the space domain expanding to the whole space R^n.

Let $G_\rho = \{x \mid |x| < \rho\}$, and $u_{0,\rho}$ and $f_{j,\rho},\ j \geqslant 1$ be the restriction of u_0 and $f_j,\ j \geqslant 1$, respectively to G_ρ. For each $\rho \geqslant 1$, we consider the following initial-boundary value problem.

$$\partial_t u_\rho = \sum_{i=1}^{n} \partial_{x_i}\big(|u_\rho|^{p-2}\partial_{x_i} u_\rho\big) + \sum_{j=1}^{\infty} f_{j,\rho}\,\frac{dB_j}{dt}, \qquad (t,x) \in (0,T) \times G_\rho, \tag{5.5}$$

$$u_\rho = 0, \qquad (t,x) \in (0,T) \times \partial G_\rho, \tag{5.6}$$

$$u_\rho(0,x) = u_{0,\rho}(x), \qquad x \in G_\rho. \tag{5.7}$$

By Theorem 2.2, we have a pathwise unique solution u_ρ such that

$$\begin{aligned} &E\bigg(\sup_{t\in[0,T]} \big\|u_\rho(t)\big\|^2_{L^2(R^n)} + \int_0^T \big\||u_\rho|^{(p-2)/2}\nabla u_\rho\big\|^2_{L^2(R^n)} dt\bigg) \\ &\quad \leqslant CE\big(\|u_0\|^2_{L^2(R^n)}\big) + C\sum_{j=1}^{\infty} E\bigg(\int_0^T \big\|f_j(t)\big\|^2_{L^2(R^n)}\,dt\bigg), \end{aligned} \tag{5.8}$$

for some positive constant C independent of ρ and T. This follows from (3.56) after u_ρ is extended to R^n such that $u_\rho = 0$ outside G_ρ. This extension is valid because of (5.6). However, this estimate is not enough for the estimate of the time derivative since we need estimates independent of ρ. We will present a necessary technical lemma.

Lemma 5.2. *Suppose $g \in L^2(R^n)$ has compact support such that $\nabla\big(|g|^{(p-2)/2}g\big) \in L^2(R^n)$. Then, it holds that*

$$\|g\|^p_{L^p(R^n)} \leqslant C\Big(\big\||g|^{(p-2)/2}\nabla g\big\|^2_{L^2(R^n)} + \|g\|^p_{L^2(R^n)}\Big) \tag{5.9}$$

and

$$\big\|\nabla\big(|g|^{p-2}g\big)\big\|^{p'}_{L^{p'}(R^n)} \leqslant C\Big(\big\||g|^{(p-2)/2}\nabla g\big\|^2_{L^2(R^n)} + \|g\|^p_{L^2(R^n)}\Big), \tag{5.10}$$

where C denotes universal positive constants.

Proof. Our major reference is [14].

We have to proceed differently depending on the space dimension n.

Case 1: $n \geqslant 3$.

$$\|g\|_{L^p(R^n)} \leqslant C\|g\|_{L^2(R^n)}^{1-\theta} \left\||g|^{(p-2)/2}\nabla g\right\|_{L^2(R^n)}^{2\theta/p}, \tag{5.11}$$

where $\theta = \frac{q(p-2)}{pq-4}$ and $\frac{1}{q} = \frac{1}{2} - \frac{1}{n}$. This yields (5.9). We also have

$$\begin{aligned}
\left\|\nabla\big(|g|^{p-2}g\big)\right\|_{L^{p'}(R^n)}^{p'} &\leqslant C\left\||g|^{(p-2)/2}\right\|_{L^{2p/(p-2)}(R^n)}^{p'} \left\||g|^{(p-2)/2}\nabla g\right\|_{L^2(R^n)}^{p'} \\
&\leqslant \left\|g\right\|_{L^2(R^n)}^{p'(1-\theta)(p-2)/2} \left\||g|^{(p-2)/2}\nabla g\right\|_{L^2(R^n)}^{p'+p'\theta(p-2)/p} \\
&\leqslant C\left(\left\||g|^{(p-2)/2}\nabla g\right\|_{L^2(R^n)}^2 + \|g\|_{L^2(R^n)}^p\right).
\end{aligned} \tag{5.12}$$

Case 2: $n = 2$.

By interpolation with help of

$$\left\|\nabla(|g|^{(p+2)/2})\right\|_{L^1(R^2)} \leqslant C\|g\|_{L^2(R^2)}\left\||g|^{(p-2)/2}\nabla g\right\|_{L^2(R^2)} \tag{5.13}$$

and

$$\|g\|_{L^{p+2}(R^2)} \leqslant C\left\|\nabla(|g|^{(p+2)/2})\right\|_{L^1(R^2)}^{2/(p+2)}, \tag{5.14}$$

we can derive (5.9). For (5.10), we see that

$$\begin{aligned}
\left\||g|^{\frac{p-2}{2}}\right\|_{L^{\frac{2p}{p-2}}(R^2)}^{p'} &= \left\|g\right\|_{L^p(R^2)}^{\frac{p(p-2)}{2(p-1)}} \\
&\leqslant \left\|g\right\|_{L^2(R^2)}^{\theta\frac{p(p-2)}{2(p-1)}} \left\|g\right\|_{L^{p+2}(R^2)}^{(1-\theta)\frac{p(p-2)}{2(p-1)}} \\
&\leqslant C\left\|g\right\|_{L^2(R^2)}^{\theta\frac{p(p-2)}{2(p-1)}+(1-\theta)\frac{p(p-2)}{(p+2)(p-1)}} \left\||g|^{\frac{p-2}{2}}\nabla g\right\|_{L^2(R^2)}^{(1-\theta)\frac{p(p-2)}{(p+2)(p-1)}},
\end{aligned} \tag{5.15}$$

where $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p+2}$, or $\theta = \frac{4}{p^2}$, and hence,

$$\begin{aligned}
\left\|\nabla(|g|^{p-2}g)\right\|_{L^{p'}(R^2)}^{p'} &\leqslant C\left\||g|^{\frac{p-2}{2}}\right\|_{L^{\frac{2p}{p-2}}(R^2)}^{p'} \left\||g|^{\frac{p-2}{2}}\nabla g\right\|_{L^2(R^2)}^{p'} \\
&\leqslant C\left\|g\right\|_{L^2(R^2)}^{\theta\frac{p(p-2)}{2(p-1)}+(1-\theta)\frac{p(p-2)}{(p+2)(p-1)}} \left\||g|^{\frac{p-2}{2}}\nabla g\right\|_{L^2(R^2)}^{p'+(1-\theta)\frac{p(p-2)}{(p+2)(p-1)}} \\
&\leqslant C\|g\|_{L^2(R^2)}^p + C\left\||g|^{\frac{p-2}{2}}\nabla g\right\|_{L^2(R^2)}^2.
\end{aligned} \tag{5.16}$$

Case 3: $n=1$.
It is easy to see that

$$\|g\|_{L^\infty(R)}^{(p+2)/2} \leqslant \|g\|_{L^2(R)} \left\| |g|^{\frac{p-2}{2}} \nabla g \right\|_{L^2(R)} \tag{5.17}$$

and thus,

$$\begin{aligned}\|g\|_{L^p(R)} &\leqslant \|g\|_{L^2(R)}^{\frac{2}{p}} \left\|g\right\|_{L^\infty(R)}^{\frac{p-2}{p}} \\ &\leqslant \|g\|_{L^2(R)}^{\frac{4}{p+2}} \left\| |g|^{\frac{p-2}{2}} \nabla g \right\|_{L^2(R)}^{\frac{2(p-2)}{p(p+2)}},\end{aligned} \tag{5.18}$$

which yields (5.9). Using these inequalities, we also find that

$$\begin{aligned}\left\|\nabla(|g|^{p-2}g)\right\|_{L^{p'}(R)}^{p'} &\leqslant C\left\|g\right\|_{L^p(R)}^{\frac{p(p-2)}{2(p-1)}} \left\||g|^{\frac{p-2}{2}}\nabla g\right\|_{L^2(R)}^{p'} \\ &\leqslant C\left\|g\right\|_{L^2(R)}^{\frac{2p(p-2)}{(p-1)(p+2)}} \left\||g|^{\frac{p-2}{2}}\nabla g\right\|_{L^2(R)}^{p'+\frac{(p-2)^2}{(p+2)(p-1)}} \\ &\leqslant C\|g\|_{L^2(R)}^{p} + C\left\||g|^{\frac{p-2}{2}}\nabla g\right\|_{L^2(R)}^{2}. \qquad \square\end{aligned} \tag{5.19}$$

We now go back to the proof of existence in Section 3. Choose any $\rho>0$ and let u_m be the approximation of u_ρ as in Section 3. It follows from (3.16)

$$\begin{aligned}&\sup_{t\in[0,T]} \left\|u_m(t)\right\|_{L^2(G_\rho)}^{p} + \left(\int_0^T \left\||u_m|^{(p-2)/2}\nabla u_m\right\|_{L^2(G_\rho)}^2 dt\right)^{p/2} \\ &\leqslant C\left\|u_m(0)\right\|_{L^2(G_\rho)}^{p} + C\sup_{t\in[0,T]}\left|\sum_{j=1}^{\infty}\int_0^t < u_m(s), f_{j,\rho}(s) > dB_j(s)\right|^{p/2} \\ &\quad + C\left(\sum_{j=1}^{\infty}\int_0^T \left\|\mathcal{P}_m f_{j,\rho}(s)\right\|_{L^2(G_\rho)}^2 ds\right)^{p/2}\end{aligned} \tag{5.20}$$

for some positive constant C independent of m, ρ, and T. By the Burkholder–Davis–Gundy inequality, we have

$$E\left(\sup_{t\in[0,T]}\left|\sum_{j=1}^{\infty}\int_0^t < u_m(s), f_{j,\rho}(s) > dB_j(s)\right|^{p/2}\right)$$

$$\leqslant CE\left(\sum_{j=1}^{\infty}\int_0^T \|u_m(t)\|^2_{L^2(G_\rho)}\|f_{j,\rho}(t)\|^2_{L^2(G_\rho)}\,dt\right)^{p/4}$$

$$\leqslant CE\left(\sup_{t\in[0,T]}\|u_m(t)\|^{p/2}_{L^2(G_\rho)}\left(\sum_{j=1}^{\infty}\int_0^T \|f_{j,\rho}(t)\|^2_{L^2(G_\rho)}\,dt\right)^{p/4}\right)$$

$$\leqslant \delta E\left(\sup_{t\in[0,T]}\|u_m(t)\|^{p}_{L^2(G_\rho)}\right)+\frac{C}{\delta}\,E\left(\sum_{j=1}^{\infty}\int_0^T \|f_{j,\rho}\|^2_{L^2(G_\rho)}dt\right)^{p/2} \tag{5.21}$$

for every $\delta > 0$, for some positive constant C independent of m, ρ and T. Hence we have

$$E\left(\sup_{t\in[0,T]}\|u_m(t)\|^{p}_{L^2(G_\rho)}\right)+E\left(\int_0^T \||u_m|^{(p-2)/2}\nabla u_m\|^2_{L^2(G_\rho)}dt\right)^{p/2}$$

$$\leqslant CE\left(\|u_m(0)\|^p_{L^2(G_\rho)}\right)+CE\left(\sum_{j=1}^{\infty}\int_0^T \|f_{j,\rho}(s)\|^2_{L^2(G_\rho)}\,ds\right)^{p/2}. \tag{5.22}$$

By the same argument as for (3.56), we can derive

$$E\left(\sup_{t\in[0,T]}\|u_\rho(t)\|^{p}_{L^2(G_\rho)}\right)+E\left(\int_0^T \||u_\rho|^{(p-2)/2}\nabla u_\rho\|^2_{L^2(G_\rho)}dt\right)^{p/2}$$

$$\leqslant CE\left(\|u_\rho(0)\|^p_{L^2(G_\rho)}\right)+CE\left(\sum_{j=1}^{\infty}\int_0^T \|f_{j,\rho}(s)\|^2_{L^2(G_\rho)}\,ds\right)^{p/2}$$

$$\leqslant CE\left(\|u_0\|^p_{L^2(R^n)}\right)+CE\left(\sum_{j=1}^{\infty}\int_0^T \|f_{j}(s)\|^2_{L^2(R^n)}\,ds\right)^{p/2}. \tag{5.23}$$

This, combined with (5.5), (5.10) and (5.23), yields

$$E\left(\left\|\frac{\partial}{\partial t}(u_\rho - W)\right\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(G_\rho))}\right)\leqslant CE\left(\|u_0\|^p_{L^2(R^n)}\right)$$

$$+CE\left(\sum_{j=1}^{\infty}\int_0^T \|f_j(s)\|^2_{L^2(R^n)}\,ds\right)^{p/2} \tag{5.24}$$

for some positive constant C independent of ρ, but depending on $T > 0$. We now extend u_ρ to R^n such that $u_\rho = 0$ outside G_ρ, and choose any $T > 0$.

Let us define

$$Q_\rho^\dagger = \sup_{t\in[0,T]} \left\| u_\rho(t) \right\|_{L^2(R^n)}^p + \left(\int_0^T \left\| |u_\rho|^{(p-2)/2} \nabla u_\rho \right\|_{L^2(R^n)}^2 dt \right)^{p/2} + \left\| \frac{\partial}{\partial t}\big(u_\rho - W\big) \right\|_{L^{p'}(0,T;W^{-1,p'}(G_\rho))}^{p'}. \tag{5.25}$$

It follows from (5.23) and (5.24) that

$$P\left(\bigcap_{L=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{\rho=k}^{\infty} \left\{ Q_\rho^\dagger \geqslant L \right\} \right) = 0. \tag{5.26}$$

To proceed from here, we need different versions of Lemmas 1.1–1.3. For $0 < N_1 < \infty$ and $0 < N_2 < \infty$, we define $\mathcal{S}_{N_1,N_2}$ to be the set of all functions v with compact support in R^n such that

$$\|v\|_{L^2(R^n)} \leqslant N_1, \qquad \left\| |v|^{\frac{p-2}{2}} \nabla v \right\|_{L^2(R^n)} \leqslant N_2. \tag{5.27}$$

By (5.9), each $v \in \mathcal{S}_{N_1,N_2}$ satisfies

$$\left\| v \right\|_{L^p(G_\rho)}^p \leqslant \left\| v \right\|_{L^p(R^n)}^p \leqslant C N_1^p + C N_2^2 \tag{5.28}$$

for all $\rho > 0$. By the same argument as for Lemma 1.1, we have

Lemma 5.3. *$\mathcal{S}_{N_1,N_2}$ is relatively compact in $L^p(G_\rho)$, for each $\rho > 0$.*

Also, by the same argument as for Lemma 1.3, we can derive the following fact.

Lemma 5.4. *Let $\{v_m\}_{m=1}^\infty$ be a sequence in $C\big([0,T]; W^{-1,p'}(G_\rho)\big) \cap L^p\big(0,T; L^p(G_\rho)\big)$ such that*

$$\|v_m\|_{L^p(0,T;L^p(G_\rho))} + \left\| |v_m|^{(p-2)/2} \nabla v_m \right\|_{L^2(0,T;L^2(G_\rho))} \leqslant N,$$

for all $m \geqslant 1$, for some positive constant N, and for each $t^* \in [0,T]$,

$$\left\| v_m(t) - v_m(t^*) \right\|_{W^{-1,p'}(G_\rho)} \to 0$$

as $t \to t^*$ in $[0,T]$, uniformly in m and t^*. Then, there is a subsequence $\{v_{m_k}\}_{k=1}^\infty$ which converges strongly in $L^p\big(0,T; L^p(G_\rho)\big) \cap C\big([0,T]; W^{-1,p'}(G_\rho)\big)$.

As in Section 3, there is a subset $\Omega^\dagger$ such that

$$\Omega^\dagger \subset \bigcup_{L=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{\rho=k}^{\infty} \Big\{ Q_\rho^\dagger \leqslant L \Big\}, \tag{5.29}$$

$$P\big(\Omega \setminus \Omega^\dagger\big) = 0 \tag{5.30}$$

and such that for each $\omega \in \Omega^\dagger$,

$$W \in C\big([0, T]; L^2(R^n)\big), \tag{5.31}$$

$$\text{Eqs. (5.5)–(5.7) hold for all } \rho \geqslant 1. \tag{5.32}$$

For each $\omega \in \Omega^\dagger$, it follows from (5.29) that there is a subsequence $\{u_{\rho_j}\}_{j=1}^{\infty}$ such that

$$Q_{\rho_j}^\dagger \leqslant L_\omega, \qquad \text{for all } j \geqslant 1, \tag{5.33}$$

for some positive integer L_ω. By virtue of Lemma 5.4 and the diagonal process, we can further extract a subsequence still denoted by $\{u_{\rho_j}\}_{j=1}^{\infty}$ such that

$$u_{\rho_j} \to u \qquad \text{strongly in } C\big([0, T]; W^{-1,p'}(G_{\rho^*})\big), \tag{5.34}$$

$$u_{\rho_j} \to u \qquad \text{weak star in } L^\infty\big(0, T; L^2(R^n)\big), \tag{5.35}$$

$$u_{\rho_j} \to u \qquad \text{strongly in } L^p\big(0, T; L^p(G_{\rho^*})\big), \tag{5.36}$$

$$u_{\rho_j}(t) \to u(t) \qquad \text{weakly in } L^2(R^n) \text{ for each } t \in [0, T] \tag{5.37}$$

and

$$|u_{\rho_j}|^{(p-2)/2} u_{\rho_j} \to |u|^{(p-2)/2} u \quad \text{weakly in } L^2\big(0, T; H^1(R^n)\big) \tag{5.38}$$

for every $\rho^* > 0$, for some function $u = u(\omega)$, which satisfies

$$\begin{aligned} &\langle u(t), \phi\rangle - \frac{1}{p-1} \int_0^t \int_G |u(s)|^{p-2} u(s)\, \Delta\phi \, dx\, ds \\ &\quad = \langle u_0, \phi\rangle + \langle \phi, W(t)\rangle, \end{aligned} \tag{5.39}$$

for all $\phi \in C_0^\infty(R^n)$ and all $t \in [0, T]$.

For the pathwise uniqueness of solutions, we will use the following fact.

Lemma 5.5. *Suppose h and g are tempered distributions over R^n, such that for some $\varepsilon > 0$,*

$$(\varepsilon - \Delta)h = g \qquad in \ R^n.$$

If $g \in W^{-1,p'}(R^n)$, then

$$\|h\|_{W^{1,p'}(R^n)} \leqslant C_\varepsilon \|g\|_{W^{-1,p'}(R^n)} \tag{5.40}$$

for some positive constant C_ε. Let $g \in L^2(R^n) \cap L^p(R^n)$. Then, we have

$$\|h\|_{L^p(R^n)} \leqslant C\varepsilon^{-\alpha} \|g\|^{\theta}_{L^2(R^n)} \|g\|^{1-\theta}_{L^p(R^n)} \tag{5.41}$$

for some positive constant C independent of $\varepsilon > 0$, where

$$\alpha = 1 - \frac{1}{p} + \frac{\theta}{p}, \qquad \theta = \frac{4}{4 + n(p-2)}.$$

Proof. Eq. (5.40) follows from the properties of the Bessel potential. For (5.41), we first assume $g \in C_0^\infty(R^n)$. It is easy to see

$$\varepsilon \|h\|_{L^2(R^n)} \leqslant \|g\|_{L^2(R^n)} \tag{5.42}$$

and

$$\begin{aligned} \varepsilon \int_{R^n} |h|^p \, dx + (p-1) \int_{R^n} |h|^{p-2} |\nabla h|^2 \, dx &\leqslant \int_{R^n} |g| \, |h|^{p-1} \, dx \\ &\leqslant \left(\int_{R^n} |g|^p \, dx \right)^{1/p} \left(\int_{R^n} |h|^p \, dx \right)^{(p-1)/p} \\ &\leqslant C\varepsilon^{-(p-1)} \int_{R^n} |g|^p \, dx + \frac{\varepsilon}{2} \int_{R^n} |h|^p \, dx. \end{aligned} \tag{5.43}$$

Hence, we have for $n \geqslant 3$,

$$\begin{aligned} \left(\int_{R^n} |h|^{pn/(n-2)} dx \right)^{(n-2)/n} &\leqslant C \int_{R^n} |h|^{p-2} |\nabla h|^2 \, dx \\ &\leqslant C\varepsilon^{-(p-1)} \int_{R^n} |g|^p \, dx, \end{aligned} \tag{5.44}$$

where C denotes positive constants independent of $\varepsilon > 0$. This yields

$$\|h\|_{L^{pn/(n-2)}(R^n)} \leqslant C\varepsilon^{-(p-1)/p}\|g\|_{L^p(R^n)}, \quad n \geqslant 3. \tag{5.45}$$

Now (5.41) follows from the interpolation of (5.42) and (5.45), and approximating g by a sequence of functions in $C_0^\infty(R^n)$.

For $n = 2$,

$$\begin{aligned} \|h\|_{L^{p+2}(R^2)} &\leqslant C\|h\|_{L^2(R^2)}^{\frac{2}{p+2}} \left\||h|^{\frac{p-2}{2}}\nabla h\right\|_{L^2(R^2)}^{\frac{2}{p+2}} \\ &\leqslant \varepsilon^{-\frac{p+1}{p+2}}\|g\|_{L^2(R^2)}^{\frac{2}{p+2}}\|g\|_{L^p(R^2)}^{\frac{p}{p+2}}. \end{aligned} \tag{5.46}$$

Again (5.41) follows from interpolation.

For $n = 1$,

$$\begin{aligned} \|h\|_{L^p(R)} &\leqslant \left\|h\right\|_{L^2(R)}^{\frac{4}{p+2}} \left\||h|^{\frac{p-2}{2}}\nabla h\right\|_{L^2(R)}^{\frac{2(p-2)}{p(p+2)}} \\ &\leqslant C\varepsilon^{-\frac{4}{p+2}-\frac{(p-1)(p-2)}{p(p+2)}}\left\|g\right\|_{L^2(R)}^{\frac{4}{p+2}}\left\|g\right\|_{L^p(R)}^{\frac{p-2}{p+2}}, \end{aligned} \tag{5.47}$$

which yields (5.41). □

Fix any $\omega \in \Omega^\dagger$. Suppose that u_1 and u_2 are solutions of (5.3)–(5.4) such that

$$u_i \in L^\infty\big(0, T; L^2(R^n)\big), \qquad i = 1, 2 \tag{5.48}$$

and

$$|u_i|^{(p-2)/2}\nabla u_i \in L^2\big(0, T; L^2(R^n)\big), \quad i = 1, 2. \tag{5.49}$$

Then, it follows from (5.9) and (5.10) that for $i = 1, 2$,

$$u_i \in L^p\big(0, T; L^p(R^n)\big) \tag{5.50}$$

and

$$\Delta(|u_i|^{p-2}u_i) \in L^{p'}\big(0, T; W^{-1,p'}(R^n)\big). \tag{5.51}$$

Since it holds that

$$\frac{\partial}{\partial t}(u_2-u_1) = \frac{1}{p-1}(\varepsilon-\Delta)\Big(|u_1|^{p-2}u_1-|u_2|^{p-2}u_2\Big) \\ -\frac{\varepsilon}{p-1}\Big(|u_1|^{p-2}u_1-|u_2|^{p-2}u_2\Big) \tag{5.51}$$

in the sense of distribution over $(0,T)\times R^n$, we find that

$$\frac{\partial}{\partial t}(u_2-u_1)\in L^{p'}\big(0,T;W^{-1,p'}(R^n)\big). \tag{5.53}$$

In the meantime, it follows from (5.41), (5.48) and (5.50) that

$$\varepsilon\int_0^T\left|\int_{R^n}\Big(|u_1|^{p-2}u_1-|u_2|^{p-2}u_2\Big)(\varepsilon-\Delta)^{-1}(u_2-u_1)\,dx\right|dt \\ \leqslant\varepsilon^{1-\alpha}M, \tag{5.54}$$

where M is a positive constant independent of ε, and $0<\alpha<1$.

By writing $v_\varepsilon=(\varepsilon-\Delta)^{-1}(u_2-u_1)$, we apply Lemma 1.3 of [13] to derive

$$\begin{aligned}&\int_0^t\left\langle\frac{\partial}{\partial t}(u_2-u_1),\ (\varepsilon-\Delta)^{-1}(u_2-u_1)\right\rangle ds\\ &\quad=\int_0^t\left\langle(\varepsilon-\Delta)\frac{\partial}{\partial t}(v_\varepsilon),\ v_\varepsilon\right\rangle ds\\ &\quad=\frac{\varepsilon}{2}\|v_\varepsilon\|^2_{L^2(R^n)}+\frac{1}{2}\|\nabla v_\varepsilon(t)\|^2_{L^2(R^n)},\quad\text{for all } t\in[0,T].\end{aligned} \tag{5.55}$$

Combining (5.54) and (5.55), we obtain

$$\varepsilon\|v_\varepsilon(t)\|^2_{L^2(R^n)}+\|\nabla v_\varepsilon(t)\|^2_{L^2(R^n)}\leqslant\varepsilon^{1-\alpha}M \tag{5.56}$$

for all $t\in[0,T]$ and all $\varepsilon>0$. Obviously, for each $t\in[0,T]$,

$$\sqrt{\varepsilon}\,v_\varepsilon\to 0\qquad\text{in } L^2(R^n) \tag{5.57}$$

and

$$\nabla v_\varepsilon\to 0\qquad\text{in } L^2(R^n) \tag{5.58}$$

as $\varepsilon \to 0$. Thus, for each $t \in [0, T]$,

$$u_2(t) - u_1(t) = \varepsilon v_\varepsilon(t) - \Delta v_\varepsilon(t) \to 0 \tag{5.59}$$

in $H^{-1}(R^n)$, as $\varepsilon \to 0$. Thus, $u_2 \equiv u_1$.

Next we consider measurability of u. As before, we set

$$\begin{aligned} Q^{\dagger}_{\rho,t^*} = {} & \sup_{t \in [0,t^*]} \|u_\rho(t)\|^p_{L^2(R^n)} + \left(\int_0^{t^*} \big\| |u_\rho|^{(p-2)/2} \nabla u_\rho \big\|^2_{L^2(R^n)} \, dt \right)^{p/2} \\ & + \left\| \frac{\partial}{\partial t}\big(u_\rho - W\big) \right\|^{p'}_{L^{p'}(0,t^*; W^{-1,p'}(G_\rho))}. \end{aligned} \tag{5.60}$$

Fix any integer $\rho^* \geqslant 1$. Let V be a closed ball in $W^{-1,p'}(G_{\rho^*})$. Then, for each fixed $t^* \in (0, T]$, we have

$$\begin{aligned} \Omega^\dagger \bigcap \big\{ \chi_{G_{\rho^*}} u(t^*) \in V \big\} = {} & \Omega^\dagger \bigcap \Bigg(\bigcup_{L=1}^{\infty} \bigcap_{v=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{\rho = k \vee \rho^*}^{\infty} \\ & \times \big\{ \chi_{G_{\rho^*}} u_\rho(t^*) \in V_v \quad \text{and} \quad Q^\dagger_{\rho,t^*} \leqslant L \big\} \Bigg), \end{aligned} \tag{5.61}$$

where $\chi_{G_{\rho^*}}$ is the characteristic function of the set G_{ρ^*}. By the same argument as above, we find that for each $\rho^* \geqslant 1$, $\chi_{G_{\rho^*}} u$ is $L^2(R^n)$-valued progressively measurable. But for each $\omega \in \Omega^\dagger$ and $t \in [0, T]$, as $\rho^* \to \infty$,

$$\chi_{G_{\rho^*}} u \to u \qquad \text{in } L^2(R^n). \tag{5.62}$$

Thus, u is also $L^2(R^n)$-valued progressively measurable. As in (3.56), we obtain

$$\begin{aligned} & E\Big(\sup_{t \in [0,T]} \|u(t)\|^p_{L^2(R^n)} \Big) + E\left(\int_0^T \big\| |u|^{(p-2)/2} \nabla u \big\|^2_{L^2(R^n)} \, dt \right)^{p/2} \\ & \quad \leqslant C E\big(\|u_0\|^p_{L^2(R^n)} \big) + C E\left(\sum_{j=1}^{\infty} \int_0^T \|f_j(t)\|^2_{L^2(R^n)} \, dt \right)^{p/2} \end{aligned} \tag{5.63}$$

for some positive constant C. This, combined with (5.3) and (5.10), yields

$$E\left(\left\|\frac{\partial}{\partial t}(u-W)\right\|^{p'}_{W^{-1,p'}(R^n)}\right) \leqslant CE\left(\|u_0\|^p_{L^2(R^n)}\right) + CE\left(\sum_{j=1}^{\infty}\int_0^T \|f_j(t)\|^2_{L^2(R^n)}\,dt\right)^{p/2}. \tag{5.64}$$

This completes the proof of Theorem 5.1.

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